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Exchange Systems, Matchings, and Transversals

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ABSTRACT

The present paper is concerned with a combinatorial structure called an exchange system. It is shown how a relation between two sets induces via matchings certain exchange systems. Applications to transversal theory are indicated, and necessary and sufficient conditions are given for a family of sets to have a transversal if some subfamily consisting of all but a finite number of the sets has a transversal.

1. INTRODUCTION

We were led to the investigation whose results are reported in this paper in attempting to find necessary and sufficient conditions in order that a family S of sets of which only a finite number are infinite have a transversal. In case S is a finite family or a family of finite sets, conditions have been known for some time now (see [3, 4, 5]). Recently R. Rado and H. A. Jung [13] treated the case in which only one of the sets is infinite. In order to extend the Rado-Jung theorem we define an "exchange system" $(E, \mathscr{I})$, where $\mathscr{I}$ is a set of subsets of E. In case the underlying set E is finite, an exchange system is what is commonly called a matroid [14]. If S is a finite family, then J. Edmonds and D. R. Fulkerson [2] showed that the set of partial transversals of S gives a matroid, a fact which can also be deduced from Theorem 1 of N. S. Mendelsohn and A. L. Dulmage [7]. In the general case the partial transversals of the family S form an exchange system and induce several other exchange systems.

Our axioms for an exchange system $(E, \mathscr{I})$ are closely related to axioms for "abstract linear dependence" as set forth by R. Rado [12] and M. N. Bleicher and G. B. Preston [1]. If the set $\mathscr{I}$ is of finite character,

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that is a set belongs to $\mathscr{I}$ if and only if every finite subset does, then our axioms are equivalent to those of Rado and Bleicher and Preston. After completing our work, a paper [9] by L. Mirsky and H. Perfect appeared in part investigating problems of a similar nature but usually in more restrictive circumstances. Most of our work is disjoint with theirs.

2. Exchange Systems

If A and B are two sets, then $A \setminus B$ is the set consisting of those elements of A which are not in B. If x is any element, then for convenience of notation the set $\{x\}$ will also be denoted by x. For $\mathscr{F}$ a set of sets and A an arbitrary set, $\mathscr{F} \cap A$ denotes the set of those sets which are intersections of members of $\mathscr{F}$ with A.

Let E be an arbitrary set and $\mathscr{I}$ a set of subsets of E. Then $(E, \mathscr{I})$ is an *exchange system* provided the following properties are satisfied:

(1) $\phi \in \mathscr{I}$.

(2) If $A \in \mathscr{I}$ and $A' \subseteq A$, then $A' \in \mathscr{I}$.

(3) (*Exchange Property*). Let $E_0 \subseteq E$ and let B_1 and B_2 be maximal members of $\mathscr{I} \cap E_0$. Let $x \in B_1 \setminus B_2$. Then there exists $y \in B_2 \setminus B_1$ such that both $(B_1 \setminus x) \cup y$ and $(B_2 \setminus y) \cup x$ are maximal members of $\mathscr{I} \cap E_0$.

It is an immediate consequence of the definition that, if $(E, \mathscr{I})$ is an exchange system and $E_0 \subseteq E$, then $(E_0, \mathscr{I} \cap E_0)$ is also an exchange system.

Theorem 1. *Let $(E, \mathscr{I})$ be an exchange system and let*

$$\mathscr{I}' = \{A \subseteq E : E \setminus A \text{ contains a maximal member of } \mathscr{I}\}.$$

If $\mathscr{I}'$ is non-empty, then $(E, \mathscr{I}')$ is an exchange system, called the dual of $(E, \mathscr{I})$.

Proof: Suppose $\mathscr{I}'$ is non-empty, or equivalently that $\mathscr{I}$ has at least one maximal member. Then properties (1) and (2) for an exchange system are satisfied by the pair $(E, \mathscr{I}')$. Hence we have only to verify property (3). Let $E_0 \subseteq E$ and let B_1' and B_2' be maximal members of $\mathscr{I}' \cap E_0$. Thus $E \setminus B_i'$ contains a maximal member B_i of $\mathscr{I}$ such that $E_0 \setminus B_i = B_i'$, $i = 1, 2$. Let $x \in B_1' \setminus B_2'$, so that $x \in B_2 \setminus B_1$. Since $(E, \mathscr{I})$ is an exchange system, there exists a $y \in B_1 \setminus B_2$ such that $(B_2 \setminus x) \cup y$ and $(B_1 \setminus y) \cup x$ are maximal members of $\mathscr{I}$. Suppose $y \notin E_0$. Then $E_0 \setminus ((B_2 \setminus x) \cup y) = E_0 \setminus (B_2 \setminus x) = B_2' \cup x$, contradicting the maximality of B_2' in $\mathscr{I}' \cap E_0$. Hence $y \in E_0$ and therefore $y \in B_2' \setminus B_1'$, so that $(B_1' \setminus x) \cup y$ and $(B_2' \setminus y) \cup x$ belong to $\mathscr{I}' \cap E_0$.

Suppose $(B_1' \setminus x) \cup y$ is not a maximal member of $\mathscr{I}' \cap E_0$ so that there exists a $z \in E_0 \setminus ((B_1' \setminus x) \cup y)$ with $(B_1' \setminus x) \cup \{y, z\}$ a member of $\mathscr{I}' \cap E_0$. Thus there exists a maximal member B_3 of $\mathscr{I}$ such that

$$E_0 \setminus B_3 \supseteq (B_1' \setminus x) \cup \{y, z\}.$$

If x were not in B_3, then $E_0 \setminus B_3 \supseteq B_1' \cup \{y, z\}$ and this contradicts the maximality of B_1' in $\mathscr{I}' \cap E_0$. Hence $x \in B_3 \setminus B_1$. Thus since $(E, \mathscr{I})$ is an exchange system, there is a $w \in B_1 \setminus B_3$ such that $(B_3 \setminus x) \cup w$ is a maximal member of $\mathscr{I}$. But then

$$E_0 \setminus ((B_3 \setminus x) \cup w)$$

properly contains B_1' (it contains $B_1' \cup y$ or $B_1' \cup z$), and this contradicts the maximality of B_1' in $\mathscr{I}' \cap E_0$. Thus $(B_1' \setminus x) \cup y$ is a maximal member of $\mathscr{I}' \cap E_0$. In an analogous way one proves that $(B_2' \setminus y) \cup x$ is also a maximal member of $\mathscr{I}' \cap E_0$. This completes the proof.

If $(E, \mathscr{I})$ is an exchange system such that $\mathscr{I}'$ is non-empty, then every member of $\mathscr{I}'$ is contained in a maximal member of $\mathscr{I}'$. Hence $(E, \mathscr{I}')$ satisfies the following stronger exchange property:

(3') Let $A \in \mathscr{I}' \cap E_0$ and let B be a maximal member of $\mathscr{I}' \cap E_0$. If $x \in A \setminus B$, then there is a $y \in B \setminus A$ such that $(B \setminus y) \cup x$ is a maximal member of $\mathscr{I}' \cap E_0$ and $(A \setminus x) \cup y$ is a member of $\mathscr{I}' \cap E_0$. If A is also maximal in $\mathscr{I}' \cap E_0$, then $(A \setminus x) \cup y$ is a maximal member of $\mathscr{I}' \cap E_0$.

It is not always the case that the dual $(E, \mathscr{I}'')$ of the dual $(E, \mathscr{I}')$ of the exchange system $(E, \mathscr{I})$ is $(E, \mathscr{I})$, but it is always true that $(E, \mathscr{I}''') = (E, \mathscr{I}')$. In case $(E, \mathscr{I}'') = (E, \mathscr{I})$, then $(E, \mathscr{I})$ is called an *inductive exchange system*. The exchange system $(E, \mathscr{I})$ is inductive if and only if every member of $\mathscr{I}$ is contained in a maximal member of $\mathscr{I}$.

If $(E, \mathscr{I})$ is an exchange system and $\mathscr{I}$ has a maximal member B which is finite, then from the exchange property it follows that all maximal members of $\mathscr{I}$ are finite with the same cardinality. In particular, if E is a finite set, then $(E, \mathscr{I})$ is a matroid and the preceding theorem reduces to Theorem 23 of Whitney [14].

3. Matchings

Let E and S be arbitrary sets and let $\imath \subseteq E \times S$ be a relation between E and S. A *matching* between E and S (with respect to the relation $\imath$)

is a relation $\iota_0 \subseteq \iota$ such that if $(a, b) \in \iota_0$ and $(a', b') \in \iota_0$, then $a = a'$ if and only if $b = b'$. If ι_0 is a matching and $(a, b) \in \iota_0$, then we write $a\iota_0 = b$. If $A = \{a : (a, b) \in \iota_0\}$ and $B = \{b : (a, b) \in \iota_0\}$, then the matching ι_0 is said to *meet* S in A and E in B, and we write $A\iota_0 = B$.

THEOREM 2. *Let E and S be arbitrary sets and ι a relation between E and S. Let $\mathscr{I}$ be the set consisting of all subsets of E which meet a matching with respect to ι. Let $\mathscr{J}$ be the set consisting of all subsets X of S such that there is a matching with respect to ι which meets E in a maximal member of $\mathscr{I}$ and meets S in a subset of $S \setminus X$. Then $(E, \mathscr{I})$ is an exchange system, and, if $\mathscr{I}$ has at least one maximal member, then $(S, \mathscr{J})$ is also an exchange system.*

PROOF: We first show $(E, \mathscr{I})$ is an exchange system. Properties (1) and (2) for an exchange system are obviously satisfied by $(E, \mathscr{I})$. We prove that $(E, \mathscr{I})$ satisfies property (3). Let $E_0 \subseteq E$ and let B_1 and B_2 be maximal members of $\mathscr{I} \cap E_0$. There exist matchings ι_1 and ι_2 which meet E_0 in B_1 and B_2, respectively. Let $x = x_0 \in B_1 \setminus B_2$ and suppose $x_0, x_1, ..., x_m$ $(m \geqslant 0)$ have been defined. If $x_m \in B_2 \setminus B_1$, we stop. Otherwise $x_m \in B_1$. If $x_m\iota_1 \notin B_2\iota_2$, then there is a matching ι_3 given by

$$\iota_3 = (\iota_2 \setminus \{(x_k, x_k\iota_2) : 1 \leqslant k < m\}) \cup \{(x_j, x_j\iota_1) : 0 \leqslant j < m\}.$$

The matching ι_3 meets E_0 in $B_2 \cup x$, contradicting the maximality of B_2. Hence there exists $x_{m+1} \in B_2$ such that $x_{m+1}\iota_2 = x_m\iota_1$. If this process does not terminate so that there is an infinite sequence of distinct elements $\{x_0, x_1, x_2, ...\}$ of E_0 such that

$$x_k\iota_1 = x_{k+1}\iota_2 \qquad (k = 0, 1, 2, ...),$$

then we define a matching ι_3 by

$$\iota_3 = \Big(\iota_2 \setminus \bigcup_{k=1}^{\infty} \{(x_k, x_k\iota_2)\}\Big) \cup \Big(\bigcup_{k=0}^{\infty} \{(x_k, x_k\iota_1)\}\Big).$$

This matching ι_3 meets E_0 in $B_2 \cup x$, contradicting once more the maximality of B_2. Thus the process must terminate. That is, there is a positive integer n such that $x_n \in B_2 \setminus B_1$. We now define matchings ι_2' and ι_1' by:

$$\iota_2' = \Big(\iota_2 \setminus \bigcup_{k=1}^{n} \{(x_k, x_k\iota_2)\}\Big) \cup \bigcup_{k=0}^{n-1} \{(x_k, x_k\iota_1)\},$$

$$\iota_1' = \Big(\iota_1 \setminus \bigcup_{k=0}^{n-1} \{(x_k, x_k\iota_1)\}\Big) \cup \bigcup_{k=1}^{n} \{(x_k, x_k\iota_2)\}.$$

Let $y = x_n$. Then the matching ι_2' meets E_0 in $(B_2 \setminus y) \cup x$ and S in $B_2\iota_2$, and the matching ι_1' meets E_0 in $(B_1 \setminus x) \cup y$ and S in $B_1\iota_1$. Hence $(B_2 \setminus y) \cup x$ and $(B_1 \setminus x) \cup y$ belong to $\mathscr{I} \cap E_0$.

Suppose $(B_2 \setminus y) \cup x$ were not a maximal member of $\mathscr{I} \cap E_0$. Hence there is a $z \in E_0$ but $z \notin (B_2 \setminus y) \cup x$ and a matching ι_3 which meets E_0 in $B_3 = (B_2 \setminus y) \cup \{x, z\}$. By replacing B_1 with B_3, x with z, and ι_1 with ι_3 in the preceding argument (the maximality of B_1 was not used there) we can conclude since $B_2 \setminus B_3 = \{y\}$ that there exist

$$z = z_0, z_1, \ldots, z_l = y$$

such that $z_1, \ldots, z_{l-1}$ are in $B_2 \cap B_3$ and

$$z_j\iota_3 = z_{j+1}\iota_2 \qquad (j = 0, 1, \ldots, l-1).$$

Since $x \in B_3 \setminus B_2$, by replacing in the first argument B_1 with B_3 and ι_1 with ι_3, there exists, since $B_2 \setminus B_3 = \{y\}$,

$$x = w_0, w_1, \ldots, w_m = y$$

such that $w_1, \ldots, w_{m-1}$ are in $B_2 \cap B_3$ and

$$w_j\iota_3 = w_{j+1}\iota_2 \qquad (j = 0, 1, \ldots, m-1).$$

But then

$$w_{m-1}\iota_3 = y\iota_2$$

and

$$z_{l-1}\iota_3 = y\iota_2,$$

so

$$w_{m-1} = z_{l-1}.$$

If $m < l$, then this implies that $x = z_{l-m}$. But $z_{l-m} \in B_2 \cap B_3$ and $x \notin B_2 \cap B_3$, a contradiction. If $l < m$, then $z = w_{m-l}$. But $w_{m-l} \in B_2 \cap B_3$ and $z \notin B_2 \cap B_3$, a contradiction. Moreover $m \neq l$, since $z \neq x$. Thus, in any case we have a contradiction and $(B_2 \setminus y) \cup \{x\}$ is a maximal member of $\mathscr{I} \cap E_0$. An analogous argument proves that $(B_1 \setminus x) \cup y$ is also a maximal member of $\mathscr{I} \cap E_0$. Thus we have proved that $(E, \mathscr{I})$ is an exchange system and, in fact, satisfies the stronger exchange property (3′).

Suppose now $\mathscr{I}$ has at least one maximal member, so that $\phi \in \mathscr{J}$. Certainly $\mathscr{J}$ satisfies property (2) for an exchange system, so that we need only verify the exchange property. Let $S_0 \subseteq S$ and let B_1 and B_2 be the maximal members of $\mathscr{J} \cap S_0$. Then there exist matchings ι_1 and ι_2

such that $\imath_k$ meets E in A_k and S in $A_k\imath_k$ where $S_0 \setminus A_k\imath_k = B_k$, and where A_k is a maximal member of $\mathscr{I}$, $k = 1, 2$. Let $x = x_0 \in B_1 \setminus B_2$. Suppose $x_0, x_1, \ldots, x_{m-1}$ are defined. If $x_{m-1} \notin A_2\imath_2$ then x_m is not defined. However, if $x_{m-1} \in A_2\imath_2$, then there exists an $e_m \in A_2$ such that $e_m\imath_2 = x_{m-1}$. If $e_m \in A_1$, then we define x_m by $x_m = e_m\imath_1$. If $e_m \notin A_1$, then we define a matching $\imath_3$ by

$$\imath_3 = (\imath_1 \setminus \{(e_k, x_k) : 1 \leqslant k < m\}) \cup \{(e_j, x_{j-1}) : 1 \leqslant j \leqslant m\}.$$

The matching $\imath_3$ meets E in $A_1 \cup e_m$, which contradicts the maximality of A_1 in $\mathscr{I}$. Hence $e_m \in A_1 \cap A_2$ and thus in this case x_m is defined. Suppose x_m is defined for all non-negative integers x_m. Then we define a matching $\imath_3$ by

$$\imath_3 = \Big(\imath_2 \setminus \bigcup_{k=0}^{\infty} \{(e_{k+1}, x_k)\}\Big) \cup \Big(\bigcup_{k=1}^{\infty} \{(e_k, x_k)\}\Big).$$

The matching $\imath_3$ meets E in A_2, a maximal member of $\mathscr{I}$, and S in $A_2\imath_3$ where $S_0 \setminus A_2\imath_3 = B_2 \cup x_0$, contradicting the maximality of B_2 in $\mathscr{J} \cap S_0$. Hence there exists an integer n such that $e_n \in A_2 \setminus A_1$, that is $x_n \notin A_1\imath_1$. Define a matching $\imath_2'$ by

$$\imath_2' = \Big(\imath_2 \setminus \bigcup_{k=0}^{n-1} \{(e_{k+1}, x_k)\}\Big) \cup \Big(\bigcup_{k=1}^{n} \{(e_k, x_k)\}\Big).$$

The matching $\imath_2'$ meets E in A_2 and meets S in $A_2\imath_2'$ where

$$S_0 \setminus A_2\imath_2' = S_0 \setminus ((A_2\imath_2 \cup x_0) \setminus x_n),$$

which equals $B_2 \cup x$ if $x_n \notin S_0$, contradicting the maximality of B_2. Hence $x_n = y$ is in S_0 and therefore in $B_2 \setminus B_1$. Then $\imath_2'$ meets S in $A_2\imath_2'$ where $S_0 \setminus A_2\imath_2' = (B_2 \setminus y) \cup x$. Thus $(B_2 \setminus y) \cup x$ is a member of $\mathscr{J} \cap S_0$. Likewise we define a matching $\imath_1'$ by

$$\imath_1' = \Big(\imath_1 \setminus \bigcup_{k=1}^{n} \{(e_k, x_k)\}\Big) \cup \Big(\bigcup_{k=0}^{n-1} \{(e_{k+1}, x_k)\}\Big).$$

The matching $\imath_1'$ meets E in A_1 and S in $A_1\imath_1'$ where $S_0 \setminus A_1\imath_1' = (B_1 \setminus x) \cup y$. Hence $(B_1 \setminus x) \cup y$ is also a member of $\mathscr{J} \cap S_0$.

Suppose $(B_2 \setminus y) \cup x$ were not a maximal member of $\mathscr{J} \cap S_0$. Then there exists a matching $\imath_3$ which meets E in A_3, a maximal member of $\mathscr{I}$, and S in $A_3\imath_3$ with $(B_2 \setminus y) \cup x$ a proper subset of $S_0 \setminus A_3\imath_3$. Let $z \in S_0 \setminus A_3\imath_3$ with $z \notin (B_2 \setminus y) \cup x$. The element z could not be y, by the maximality of B_2, so that $z \notin B_2 \cup x$.

By replacing in the first argument B_1 with $(B_2 \setminus y) \cup \{x, z\}$, x with z, and ι_1 with ι_3 (the maximality of B_1 was not used there), then since $B_2 \setminus ((B_2 \setminus y) \cup \{x, z\}) = \{y\}$, there exist

$$z = z_0, z_1, \ldots, z_l = y$$

and

$$e_1, e_2, \ldots, e_l, \quad \text{in} \quad E$$

such that

$$e_j \iota_2 = z_{j-1} \qquad (j = 1, 2, \ldots, l)$$

and

$$e_j \iota_3 = z_j \qquad (j = 1, 2, \ldots, l)$$

with

$$\{z, \ldots, z_{l-1}\} \subseteq A_2 \iota_2 \cap A_3 \iota_3 .$$

Similarly by replacing in the first argument B_1 with $(B_2 \setminus y) \cup \{x, z\}$ and ι_1 with ι_3, then there exist

$$x = w_0, w_1, \ldots, w_m = y$$

and

$$f_1, f_2, \ldots, f_m \quad \text{in} \quad E$$

such that

$$f_j \iota_2 = w_{j-1} \qquad (j = 1, 2, \ldots, m)$$

and

$$f_j \iota_3 = w_j \qquad (j = 1, 2, \ldots, m)$$

with

$$\{w_1, \ldots, w_{m-1}\} \subseteq A_2 \iota_2 \cap A_3 \iota_3 .$$

But then

$$e_l \iota_3 = y = f_m \iota_3 .$$

Thus $e_l = f_m$. If $m < l$, this implies $f_1 = e_{l-m}$ so that

$$x = f_1 \iota_2 = e_{l-m-1} \iota_3 \in A_3 \iota_3 ,$$

which is a contradiction since $x \in (B_2 \setminus y) \cup \{x, z\} \subseteq S_0 \setminus A_3 \iota_3$. If $l < m$, this implies $z \in A_3 \iota_3$, a similar contradiction. If $l = m$, then $x = z$, a contradiction. Hence $(B_2 \setminus y) \cup x$ is a maximal member of $\mathscr{I} \cap S_0$. In an analogous way one proves that $(B_1 \setminus x) \cup y$ is also a maximal member of $\mathscr{J} \cap S_0$. Therefore $(S, \mathscr{J})$ is an exchange system and indeed satisfies the stronger exchange property (3′). This completes the proof of the theorem.

Let $\imath$ be a relation between the sets E and S. By Theorem 2 and the corresponding theorem with the roles of E and S interchanged, $\imath$ induces the exchange systems:

(1) $(E, \mathscr{I}_E)$, $(S, \mathscr{I}_S)$, which are always defined.

(2) $(E, \mathscr{J}_E)$ which is defined when $\mathscr{I}_S$ has maximal members. $(S, \mathscr{J}_S)$ which is defined when $\mathscr{I}_E$ has maximal members.

(3) $(E, \mathscr{I}'_E)$, the dual of $(E, \mathscr{I}_E)$, which is defined when $\mathscr{I}_E$ has maximal members.
$(S, \mathscr{I}'_S)$, the dual of $(S, \mathscr{I}_S)$, which is defined when $\mathscr{I}_S$ has maximal members.

(4) $(E, \mathscr{J}'_E)$, the dual of $(E, \mathscr{J}_E)$, which is defined when $\mathscr{J}_E$ has maximal members.
$(S, \mathscr{J}'_S)$, the dual of $(S, \mathscr{J}_S)$, which is defined when $\mathscr{J}_S$ has maximal members.

From the definitions it follows that if $(E, \mathscr{J}'_E)$ is defined then $\mathscr{J}'_E \subseteq \mathscr{I}_E$; in fact members of $\mathscr{J}'_E$ are subsets of members X of $\mathscr{I}_E$ which are minimal with respect to the property that there exists a matching which meets E in X and S in a maximal member of $\mathscr{I}_S$. Moreover $\mathscr{J}'_E = \mathscr{I}_E$ if and only if every member of $\mathscr{I}_E$ satisfies the above property. In particular, if E is finite, then $\mathscr{J}'_E = \mathscr{I}_E$ (and thus, $\mathscr{J}_E = \mathscr{I}'_E$) and $\mathscr{J}'_S = \mathscr{I}_S$ (and thus $\mathscr{J}_S = \mathscr{I}'_S$).

Let $\imath_1$ and $\imath_2$ be matchings meeting E, respectively, in $B_1 \in \mathscr{I}_E$ and B_2 a maximal member of $\mathscr{I}_E$. Let $X \subseteq B_1 \setminus B_2$. Then it follows from the proof of Theorem 2 that for each $x \in X$ there exists a sequence

$$x = x_0, x_1, \ldots, x_m = y_x$$

such that

$$x_k \imath_1 = x_{k+1} \imath_2 \qquad (k = 0, 1, \ldots, m-1).$$

If $x' \neq x$ is also in X, the sequence corresponding to x' is disjoint from the sequence corresponding to x. Thus we can "exchange" $Y_X = \bigcup_{x \in X} \{y_x\}$ for X in B_2. That is, there is a matching, $\imath'_2$ meeting E in $(B_2 \setminus Y_X) \cup X = B'_2$ and S in $B_2 \imath_2$. However, B'_2 need not be a maximal member of $\mathscr{I}_E$ even if B_1 and B_2 both are, as the example below shows. Note that if B_1 and B_2 are maximal and $X = B_1 \setminus B_2$, then $Y_X = B_2 \setminus B_1$.

EXAMPLE. Let E be the set of positive integers. We define a set S of subsets $\{S_1, S_2, S_3, \ldots\}$ of E by

$$S_{3k+1} = \{5k+1, 5k+2\} \qquad (k = 0, 1, 2, \ldots),$$
$$S_{3k+2} = \{5k+2, 5k+3, 5k+4\} \qquad (k = 0, 1, 2, \ldots),$$
$$S_{3k+3} = \{5k+4, 5k+5, 5k+6\} \qquad (k = 0, 1, 2, \ldots).$$

We define a relation ι between E and S by

$$e\iota S_k \quad \text{if and only if} \quad e \in S_k.$$

Define $\iota_1 \subseteq \iota$ by

$$\begin{array}{ll} 1\iota_1 S_1 & \\ 3\iota_1 S_2 & \\ (5k+6)\, \iota_1 S_{3k+3} & (k = 0, 1, 2, \ldots) \\ (5k+7)\, \iota_1 S_{3k+4} & (k = 0, 1, 2, \ldots) \\ (5k+8)\, \iota_1 S_{3k+5} & (k = 0, 1, 2, \ldots). \end{array}$$

Then ι_1 is a matching meeting E in B_1, which is a maximal member of $\mathscr{I}_E$. We also define $\iota_2 \subseteq \iota$ by

$$\begin{array}{ll} 2\iota_2 S_1 & \\ (5k+4)\, \iota_2 S_{3k+2} & (k = 0, 1, 2, \ldots) \\ (5k+5)\, \iota_2 S_{3k+3} & (k = 0, 1, 2, \ldots) \\ (5k+6)\, \iota_2 S_{3k+4} & (k = 0, 1, 2, \ldots). \end{array}$$

Also ι_2 is a matching meeting E in B_2, a maximal member of $\mathscr{I}_E$. Let $X = \{5k+7 : k = 0, 1, 2, \ldots\} \subseteq B_1 \setminus B_2$. Then for $k = 0, 1, 2, \ldots$

$$(5k+7)\, \iota_1 S_{3k+4}, \quad (5k+6)\, \iota_2 S_{3k+4}, \quad (5k+6)\, \iota_1 S_{3k+3}, \quad (5k+5)\, \iota_2 S_{3k+3}.$$

Then $Y_X = \{5k+5 : k = 0, 1, 2, \ldots\} \subseteq B_2 \setminus B_1$ and

$$\begin{aligned} B' &= (B_2 \setminus Y_X) \cup X \\ &= \{2\} \cup \{5k+4 : k = 0, 1, 2, \ldots\} \cup \{5k+7 : k = 0, 1, 2, \ldots\} \\ &\quad \cup \{5k+6 : k = 0, 1, 2, \ldots\}, \end{aligned}$$

with $B' \in \mathscr{I}_E$. But B' is not maximal in $\mathscr{I}_E$, for ι'', defined below, meets E in $B'' = B' \cup 1$:

$$\begin{array}{ll} (5k+1)\, \iota'' S_{3k+1} & (k = 0, 1, 2, \ldots) \\ (5k+2)\, \iota'' S_{3k+2} & (k = 0, 1, 2, \ldots) \\ (5k+4)\, \iota'' S_{3k+3} & (k = 0, 1, 2, \ldots). \end{array}$$

THEOREM 3. *Let $\imath$ be a relation between E and S. Let $\imath_1$ and $\imath_2$ be matchings meeting E in maximal members B_1 and B_2 of $\mathscr{I}_E$, respectively. Then there exists a matching $\imath_1'$ which meets E in B_1 and S in $B_2\imath_2$. Likewise let B_3 and B_4 be maximal members of $\mathscr{I}_E$ so that there are matchings $\imath_3$ and $\imath_4$ meeting E in $E \setminus B_3$ and $E \setminus B_4$, respectively. Then there exists a matching $\imath_3'$ meeting E in $E \setminus B_3$ and S in $(E \setminus B_4)\,\imath_4$.*

PROOF: Let $X = B_1 \setminus B_2$. Then Y_X, as defined in the previous discussion, equals $B_2 \setminus B_1$. Hence there is a matching $\imath_1'$ which meets E in $(B_2 \setminus Y_X) \cup X = B_1$ and S in $B_2\imath_2$. In an analogous way taking $X = B_3 \setminus B_4$ then $Y_X = B_4 \setminus B_3$ and there exists a matching $\imath_3'$ meeting E in $(E \setminus B_3)$ and S in $(E \setminus B_4)\,\imath_4$.

COROLLARY. *Maximal members of $\mathscr{I}_E$ have the same cardinality.*

THEOREM 4. *Let E and S be sets and let $\imath_1$ and $\imath_2$ be matchings meeting E in X_1 and X_2 and S in Y_1 and Y_2, respectively. Let $\imath$ be the relation between E and S defined by $\imath = \imath_1 \cup \imath_2$. Then there is a matching $\imath_3 \subseteq \imath$ such that $\imath_3$ meets E in $X_1' \supseteq X_1$ and S in $Y_2' \supseteq Y_2$. If $\imath_4$ is any matching, $\imath_4 \subseteq \imath$, meeting E in $X_1 \supseteq X_1$ and S in $Y_2'' \supseteq Y_2$, then $X_1'' \subseteq X_1'$, and $Y_2'' \subseteq Y_2'$.*

PROOF: Let $(E, \mathscr{I}_E)$ be the exchange system induced by the relation $\imath$. Then $X_2 \in \mathscr{I}_E$. Let $x \in X_1 \setminus X_2$ and set $x_0 = x$. Suppose $x_0, x_1, \ldots, x_{m-1}$ are defined with $x_{m-1}\imath_1 \in Y_1 \cap Y_2$. Then we define x_m by $x_m\imath_2 = x_{m-1}\imath_1$. If x_m is defined for $m = 0, 1, 2, \ldots$, then as before there is a matching which meets E in $X_2 \cup x$ and S in Y_2. If x_{m+1} is not defined, then either $x_m \in X_2 \setminus X_1$ or else $x_m\imath_1 \in Y_1 \setminus Y_2$. If $x_m\imath_1 \in Y_1 \setminus Y_2$, then again there is a matching which meets E in $X_2 \cup x$ and S in $Y_2 \cup x_m\imath_1$. We may do this for each $x \in X_1 \setminus X_2$ and set

$$X_2' = X_2 \cup \{x \in X_1 \setminus X_2 : x_m \notin X_2 \setminus X_1, \quad m = 0, 1, 2, \ldots\}.$$

Then there exists a unique matching $\imath_2' \subseteq \imath$ which meets E in X_2'. Since any $x \in X_1 \setminus X_2'$ satisfies $x\imath_1 \in X_2'\imath_3'$, the uniqueness of $\imath_2'$ implies the maximality of X_2' in $\mathscr{I}_E$. Let $Y_2' = X_2'\imath_2'$. Then since $\imath_2''$ is unique, Y_2' is maximal in S with respect to meeting a matching which meets E in X_2'. Similarly we find X_1' a maximal member of $\mathscr{I}_E$ containing X_1. By Theorem 3 there is a matching which meets E in X_2' and S in Y_2', which was to be proved.

Theorem 4 contains the principal theorem proved by H. Perfect and J. S. Pym in [10]. They established the existence of X_1' and X_2' satisfying the first statement of Theorem 4 but not the maximality statement. Of course, Theorem 4 contains the Schröder-Bernstein theorem [6].

Let $S = (S_\nu : \nu \in I)$ be a family of subsets of a set indexed by I. Let ι be the relation between E and I defined by

$$e \iota \nu \quad \text{if and only if} \quad e \in S_\nu .$$

Then the set $\mathscr{I}_E$ (defined relative to this relation ι) is the set of *partial transversals* of the family S. A subfamily $S' = (S_\nu : \nu \in I')$ of S which has a transversal (that is, $I' \in \mathscr{I}_I$) is called *representable*. Since a finite exchange system is a matroid, Theorem 2 extends the first theorem of Edmonds and Fulkerson [2] that the partial transversals of a finite family of finite sets are the independent sets of a matroid. Theorem 5.2 of Mirsky and Perfect [9] is the special case of the first part of our Theorem 2 when each element of E belongs to only finitely many sets of the family S.

When S is a family of subsets of E, the family $\mathscr{I}'_E$ consists of the partial transversals which are contained in a minimal transversal of a maximal representable subfamily of S. Theorem 3 says that *if B is a transversal of one maximal representable subfamily of S, then B is a transversal of every maximal representable subfamily of S.* Likewise a maximal (minimal) transversal of a maximal representable subfamily of S is a maximal (minimal) transversal of every maximal representable subfamily of S. Perfect and Pym's Theorem 6 [10] states that any two maximal representable subfamilies of S have *some* common transversal. Also Mirsky and Perfect's Theorem 3.4 [9] is a much weaker result than ours.

4. Transversals

Let $S = (S_\nu : \nu \in I)$ be a family of subsets of a set E indexed by I. If either I is finite or each S_ν is a finite set, necessary and sufficient conditions for the family S to be representable are that for $k = 1, 2, 3,...$ and for all $\nu_1, \nu_2, ..., \nu_k$ with $1 \leqslant \nu_1 < \nu_2 < \cdots < \nu_k$ the set $S_{\nu_1} \cup S_{\nu_2} \cup \cdots \cup S_{\nu_k}$ contain at least k elements (Hall's condition). In case $|I| < \infty$, this was proved by P. Hall [3] while the other case was first proved by M. Hall Jr. [4, 5]. If $(E, \mathscr{I})$ is an exchange system of finite character, then assuming that I is finite or each S_ν is finite, R. Rado [11, 12] proved that some member of $\mathscr{I}$ is a transversal of the family S if and only if each $S_{\nu_1} \cup S_{\nu_2} \cup \cdots \cup S_{\nu_k}$ contains a member of $\mathscr{I}$ of cardinality at least k.

If $A = (A_i : i \in I)$ and $B = (B_j : j \in J)$ are two families of subsets of E where we assume without loss of generality that $I \cap J = \phi$, then we define a new family $A \oplus B = (C_k : k \in I \cup J)$ where $C_k = A_k$ or B_k according as $k \in I$ or $k \in J$. We identify a family of one set with the set.

The problem of finding necessary and sufficient conditions in order that an infinite family of arbitrary sets have a transversal is one of the outstanding unsettled questions in transversal theory. R. Rado and H. A. Jung [13] settled the case when only one set is infinite as follows: Let $S = (S_\nu : \nu \in I) \oplus T$ with each S_ν a finite set and suppose $(S_\nu : \nu \in I)$ satisfies Hall's condition. Let C be the interscetion of all transversals of $(S_\nu : \nu \in I)$. Then S has a transversal if and only if $T \setminus C \neq \phi$. Moreover they determine what C is. The following theorem extends the Rado-Jung theorem to the case of finitely many infinite sets.

THEOREM 5. *Let $S = (S_\nu : \nu \in I) \cup (T_1, T_2, \ldots, T_n)$ be a family of subsets of a set E. Let $S^0 = (S_\nu : \nu \in I_0)$ be any maximal representable subfamily of $(S_\nu : \nu \in I)$. Then the family $(S_\nu : \nu \in I_0) \cup (T_1, T_2, \ldots, T_n)$ is representable if and only if the following condition is satisfied:*

For any distinct $a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}} \in T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k} \setminus C(E)$

such that $a_{i_j} \notin C(E \setminus \{a_{i_\nu} : \nu < j\})$ $\quad (1 \leqslant j \leqslant k-1)$,

$$(*) \quad (T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k} \setminus \{a_{i_\nu} : \nu < k\}) \setminus C(E \setminus \{a_{i_\nu} : \nu < k\}) \neq \phi$$
$$(1 \leqslant k \leqslant n)$$
$$(1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n).$$

Here $C(E_0)$ for $E_0 \subseteq E$ is the intersection of all transversals of S^0 contained in E_0. (If there are no such transversals, $C(E_0) = E_0$.)

PROOF: Define a relation ι between E and I by

$$e \iota \nu \quad \text{if and only if} \quad e \in S_\nu \qquad (e \in E, \nu \in I).$$

The family I^0 is a maximal member of $\mathscr{I}_I$, so that the exchange system $(E, \mathscr{I}_E)$ is defined and by Theorem 3 $\mathscr{I}_E$ consists of the subsets of all complements of transversals of the family S^0.

Suppose $S^0 \oplus (T_1, T_2, \ldots, T_n)$ has a transversal

$$\{e_\nu : \nu \in I_0\} \cup \{t_1, t_2, \ldots, t_n\}.$$

Let $1 \leqslant k \leqslant n$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Also choose $a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}$ in $T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k} \setminus C(E)$, with

$$a_{i_j} \notin C(E \setminus \{a_{i_\nu} : \nu < j\}), \qquad 1 \leqslant j \leqslant k$$

$(t_{i_1}, t_{i_2}, \ldots, t_{i_{k-1}}$ satisfy this property so that such a_{i_j}'s exist). Let

$$E_0 = \{t_{i_1}, t_{i_2}, \ldots, t_{i_k}\} \cup \{a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}\}.$$

Then $\{a_{i_1}, a_{i_2},..., a_{i_{k-1}}\} \in \mathscr{I}_E \cap E_0$, since

$a_{i_1} \notin C(E)$ implies $\{a_{i_1}\} \in \mathscr{I}_E \cap E_0$,

$a_{i_2} \notin C(E \setminus \{a_{i_1},\})$ now implies $\{a_{i_1}, a_{i_2}\} \in \mathscr{I}_E \cap E_0$,

...,

$a_{i_{k-1}} \notin C(E \setminus \{a_{i_1},..., a_{i_{k-2}}\})$ now implies $\{a_{i_1},..., a_{i_{k-1}}\} \in \mathscr{I}_E \cap E_0$.

Since $\{t_{i_1}, t_{i_2},..., t_{i_k}\}$ is a member of $\mathscr{I}_E \cap E_0$ of cardinality k, the exchange property in $(E_0, \mathscr{I}_E \cap E_0)$ implies that there is a member of $\mathscr{I}_E \cap E_0$ of cardinality at least k, containing $\{a_{i_1}, a_{i_2},..., a_{i_{k-1}}\}$. Since $k - 1 < k$, there is a

$$t_0 \in \{t_{i_1},..., t_{i_k}\} \subseteq T_{i_1} \cup \cdots \cup T_{i_k} \setminus \{a_{i_1}, a_{i_2},..., a_{i_{k-1}}\}$$

such that

$$t_0 \notin C(E \setminus \{a_{i_1},..., a_{i_{k-1}}\}).$$

Thus property (*) is satisfied.

Conversely, suppose condition (*) is satisfied. Let $1 \leqslant k \leqslant n$ and $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. Condition (*) implies that $T_{i_1} \setminus C(E) \neq \phi$. Let a_{i_1} be any element of $T_{i_1} \cup T_{i_2} \setminus C(E)$. Condition (*) now implies that

$$(T_{i_1} \cup T_{i_2} \setminus \{a_{i_1}\}) \setminus C(E \setminus \{a_{i_1}\}) \neq \phi.$$

Let a_{i_2} be any element of $T_{i_1} \cup T_{i_2} \cup T_{i_3} \setminus C(E)$ different from a_{i_1} such that $a_{i_2} \notin C(E \setminus \{a_{i_1}\})$. As above $\{a_{i_1}, a_{i_2}\} \in \mathscr{I}_E$. Continuing like this, we obtain distinct elements $a_{i_1}, a_{i_2},..., a_{i_k}$ in $T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k}$ with $\{a_{i_1}, a_{i_2},..., a_{i_k}\} \in \mathscr{I}_E$. Thus the family $(T_1, T_2,..., T_n)$ has the property that the union of any k of the sets contains a member of $\mathscr{I}_E$ of cardinality at least k. Applying R. Rado's theorem [11], which was quoted previously, we obtain distinct elements $t_1, t_2,..., t_n$ with $t_i \in T_i$, $1 \leqslant i \leqslant n$, and $\{t_1, t_2,..., t_n\} \in \mathscr{I}_E$. Thus $E \setminus \{t_1, t_2,..., t_n\}$ contains a transversal of the family S^0. Hence $S^0 \oplus (T_1, T_2,..., T_n)$ has a transversal. This finishes the proof.

In case the family S itself has a transversal and $n = 1$ then Theorem 5 is the Rado-Jung theorem quoted above, since condition (*) reduces to $T_1 \setminus C(E) \neq \phi$. As is evident from the proof, the essence of condition (*) is that $T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_k}$ contains a set of k elements which is in the complement of *some* transversal of the family S^0. The importance of the theorem is that to show that the family $S^0 \cup (T_1, T_2,..., T_n)$ has a transversal one need only find *some* set $a_{i_1}, a_{i_2},..., a_{i_{k-1}}$ satisfying the indicated properties.

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